

SPARSE SPANNING k -CONNECTED SUBGRAPHS IN TOURNAMENTS

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ABSTRACT. In 2009, Bang-Jensen asked whether there exists a function $g(k)$ such that every strongly k -connected n -vertex tournament contains a strongly k -connected spanning subgraph with at most $kn + g(k)$ arcs. In this paper, we answer the question by showing that every strongly k -connected n -vertex tournament contains a strongly k -connected spanning subgraph with at most $kn + 750k^2 \log(k + 1)$ arcs.

1. INTRODUCTION

Search of certain subgraphs which inherit the properties of the original graph has a long history. For example, Hajnal [6] and Thomassen [14] proved that a graph G with high enough connectivity has two vertex disjoint k -connected subgraphs which together cover all vertices. Also Thomassen [13] made a conjecture that a graph G with high enough connectivity has a k -connected spanning bipartite subgraph.

For directed graphs, such problems become more difficult. One of most important problems in this direction is the following *MSSS_k problem*, where MSSS_k stands for Minimum Spanning Strongly k -connected Subgraph: for given strongly k -connected digraph D , find a spanning strongly k -connected subgraph of D with as few arcs as possible. For $k = 1$, we call it *MSSS problem* by omitting k . It is known that the Hamilton cycle problem can be solved if one can solve the MSSS problem. Thus MSSS problem is a generalization of Hamilton cycle problem, so it has been studied extensively (see e.g [1, 2] for a survey). Since Hamilton cycle problem is NP-hard for general directed graphs, MSSS problem is also NP-hard for general directed graphs. Thus it makes sense to consider subclasses of directed graphs for this problem, and this problem is solvable in polynomial-time for several classes of graphs including the class of all tournaments. (see [3, 4]) However, it is not known whether MSSS_k problem is solvable in polynomial-times for tournaments for $k \geq 2$.

Naturally, one can ask about the size (the number of arcs) of minimum spanning strongly k -connected subgraphs for strongly k -connected tournaments. If we consider the same question for arc-connectivity, the following theorem was proved by Bang-Jensen, Huang and Yeo in 2004.

Theorem 1.1. [5] *For $k \geq 1$, every strongly k -arc-connected n -vertex tournament contains a strongly k -arc-connected spanning subgraph D with $|E(D)| \leq nk + 136k^2$.*

This gives us an upper bound of the number of arcs in minimum spanning strongly k -arc-connected subgraphs for strongly k -arc-connected tournaments. However, for vertex-connectivity, no good upper bound was known. Indeed, Bang-Jensen [1] asked the following question in 2009.

Question 1.2. [1] *For $k \geq 1$, does there exist a function $g = g(k)$ such that every strongly k -connected n -vertex tournament has a strongly k -connected spanning subgraph with at most $kn + g(k)$ arcs?*

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In this paper, we answer this question by proving the following theorem.

Theorem 1.3. *For $k \geq 1$, every strongly k -connected tournament T with n vertices has a strongly k -connected spanning subgraph D with at most $kn + 750k^2 \log(k+1)$ arcs.*

Thus $g(k) = 750k^2 \log(k+1)$ is sufficient for answering Question 1.2. This is asymptotically best possible up to logarithmic factor. One of two main ingredients for the proof of Theorem 1.3 is Lemma 3.4 which is, roughly speaking, a tool guaranteeing a sparse linkage structure from/to certain vertex-sets for any tournament. The other main ingredient is “robust linkage structures” introduced by Kühn, Lapinskas, Osthus and Patel in [8] to prove a conjecture of Thomassen on edge-disjoint Hamilton cycles in highly connected tournaments. Robust linkage structure is a very powerful tool for studying highly connected tournament. Further results were obtained by this method [7, 9, 11, 12]. The novelty of the proof of Theorem 1.3 is that it produces a highly connected ‘sparse’ subgraph in the tournament, whereas previous applications of the method only produced highly connected very dense subgraphs (To be more precise, they produced complete tournaments or complete bipartite oriented graphs).

2. BASIC TERMINOLOGY AND TOOLS

For a positive integer $N \geq 1$, $[N]$ denotes the set $\{1, \dots, N\}$. Let $\log := \log_2$. A *graph* or *simple graph* is an undirected graph without multiple edges between two vertices and loops. A *directed graph* or *digraph* $D = (V, E)$ is a pair of a vertex set $V(D) = V$ and an arc set $E(D) = E$, where E is a collection of ordered pairs in $V \times V$. We let \vec{uv} denote $(u, v) \in V \times V$ an *arc from u to v* . An *oriented graph* is a digraph obtained by orienting each edge $e \in E(G)$ for a simple graph G . An n -vertex *tournament* is an oriented graph obtained by orienting each edge $e \in E(K_n)$, where K_n is a simple complete graph of order n . For a set S of vertices, $D - S$ denote the induced digraph $D[V(D) \setminus S]$. For a set E' of arcs, $D - E'$ denote the digraph $(V(D), E(D) \setminus E')$. We say a digraph $D' = (V', E')$ is a *subgraph of $D = (V, E)$* if $V' \subseteq V$ and $E' \subseteq E$. We denote $D' \subseteq D$ if D' is a subgraph of D .

For a collection of arcs E , we let $V(E) := \{u : \exists v \text{ such that } \vec{uv} \in E \text{ or } \vec{vu} \in E\}$. A *path* always denotes a directed path. A path $P = (v_1, v_2, \dots, v_n)$ is called a *path from v_1 to v_n* , and we say v_i is the i -th vertex of P . Sometimes, we consider the path P as a collection of arcs and $V(P)$ denotes $\{v_1, \dots, v_n\}$. A directed graph $D = (V, E)$ is *strongly connected* if for any $u, v \in V$, there is a path from u to v . We say that digraph D is *strongly k -connected*, if $|V| \geq k+1$ and for $S \subseteq V$ with $|S| \leq k-1$, the digraph $D - S$ remains strongly connected. Similarly, D is *strongly k -arc-connected*, if for $W \subseteq E$ with $|W| \leq k-1$, the digraph $D - W$ remains strongly connected. It is easy to see that every strongly k -connected digraph is strongly k -arc-connected. For a directed graph $D = (V, E)$ and $v \in V$, let

$$N_D^+(v) := \{u \in V(D) : \vec{vu} \in E(D)\} \text{ and } N_D^-(v) := \{u \in V(D) : \vec{uv} \in E(D)\}.$$

We call u an *out-neighbor* of v if $\vec{vu} \in E(D)$ and u an *in-neighbor* of v if $\vec{uv} \in E(D)$. We define

$$d_D^+(v) := |N_D^+(v)|, \quad d_D^-(v) := |N_D^-(v)|, \quad d_D^0(v) := d_D^+(v) + d_D^-(v), \\ \delta^+(D) = \min_{v \in V(D)} d^+(v), \quad \delta^-(D) = \min_{v \in V(D)} d^-(v) \text{ and } \delta^0(D) = \min_{v \in V(D)} d^0(v).$$

For a digraph D , $B \subseteq V(D)$ *out/in-dominates* $C \subseteq V(D)$ if every vertex in C is an out/in-neighbor of a vertex in B , respectively. A tournament T is *transitive* if $V(T)$ can be ordered into v_1, \dots, v_n such that $\vec{v_i v_j} \in E(T)$ if and only if $i < j$. We say a directed path $P = (v_1, \dots, v_p)$ in T is *backwards-transitive* if $\vec{v_i v_j} \in E(T)$ whenever $i \geq j+2$. For a vertex v and a vertex-set $U = \{u_1, \dots, u_k\}$, we say a collection $\{P_1, \dots, P_k\}$ of k paths a *k -fan from v to U* if P_i is a path from v to $u_i \in U$, $U \cap V(P_i) = \{u_i\}$ for each $i \in [k]$, and $V(P_i) \cap V(P_j) = \{v\}$ for distinct $i, j \in [k]$. Similarly, we say a collection $\{P_1, \dots, P_k\}$ of k paths a *k -fan from U to v* if P_i is a path from $u_i \in U$ to v , $U \cap V(P_i) = \{u_i\}$ for each $i \in [k]$, and $V(P_i) \cap V(P_j) = \{v\}$ for distinct $i, j \in [k]$.

We will use the following well-known fact deduced from Menger's theorem later. We omit the proof.

Fact 1. For strongly k -connected digraph D , a vertex $v \in V(D)$ and $U \subseteq V(D)$ with $|U| \geq k$, a k -fan from v to U and a k -fan from U to v always exist.

Lemma 2.1. For positive integers $n \geq 2$ and k , an n -vertex tournament T has at least k vertices v with $d_T^+(v) \geq (n-k)/2$. Moreover, T has a vertex v with $n/4 \leq d_T^+(v) \leq 3n/4$ and a vertex u with $n/4 \leq d_T^-(u) \leq 3n/4$.

Proof. Note that any n -vertex tournament contains a vertex with out-degree at least $(n-1)/2$. Let v_1, \dots, v_n be the vertices of T such that $d_T^+(v_1) \geq \dots \geq d_T^+(v_n)$. Then $T[\{v_k, \dots, v_n\}]$ contains a vertex with out-degree at least $(n-k)/2$, thus $d_T^+(v_k) \geq (n-k)/2$. Hence T contains k vertices of out-degree at least $(n-k)/2$. Also, this gives us at least $n/2$ vertices with out-degree at least $n/4$, and at least $n/2 + 1$ vertices with in-degree at least $n/4 - 1$. Hence there exists a vertex v with $n/4 \leq d_T^+(v) \leq (n-1) - (n/4 - 1) = 3n/4$. Same argument with the tournament with reverse orientation also gives us a vertex u with $n/4 \leq d_T^-(u) \leq 3n/4$. \square

For a transitive tournament T and ordering $\sigma = (v_1, \dots, v_s)$, we say T is a transitive tournament with respect to σ if $\overrightarrow{v_i v_j} \in E(T)$ whenever $i < j$.

Lemma 2.2. Let v be a vertex in an n -vertex tournament T with $d_T^+(v) = d$. Then there exist $A \subseteq V(T)$ and a vertex $a \in A$ such that the following properties hold:

- (a1) We have $\frac{1}{2} \log(d+1) + 1 \leq s \leq \frac{5}{2} \log(d+1) + 2$ where $s = |A|$.
- (a2) $T[A]$ is a transitive tournament with respect to the ordering (v_1, \dots, v_s) with source v and sink a .
- (a3) A in-dominates $V(T) \setminus A$
- (a4) For $1 \leq i \leq s/5 - 12$, we have

$$|N_T^+(v_i) \setminus A|, |N_T^-(v_i) \setminus A| \geq 8d^{1/7} - 1.$$

- (a5) For $1 \leq i \leq s - 5 \log(k) - 30$, we have

$$|N_T^+(v_i) \setminus A|, |N_T^-(v_i) \setminus A| \geq 1000k^2.$$

Proof. If $d = 0$, then let $L_1 = \emptyset$ and $A := \{v_1\}$. Then it is obvious that A with an ordering (v_1) satisfies all (a1)–(a5). Now suppose $d \geq 1$. Let $v_1 := v$, $A_1 := \{v_1\}$ and $L_1 := N_T^+(v_1)$. Suppose L_1, \dots, L_i has already been defined with $|L_i| \geq 1$. If L_i contains only one vertex u , let $v_{i+1} := u$ and $A_{i+1} := A_i \cup \{v_{i+1}\}$. If $|L_i| \geq 2$, Lemma 2.1 implies that there exists a vertex $u \in L_i$ with $|L_i|/4 \leq d_{T[L_i]}^+(u) \leq 3|L_i|/4$. Let $v_{i+1} := u$ and $L_{i+1} := L_i \cap N_T^+(v_{i+1})$. This procedure gives vertices v_1, \dots, v_s and sets L_1, \dots, L_s with $L_s = \emptyset$. We let $A := A_s$ with ordering (v_1, \dots, v_s) and let $a := v_s$. From the construction, (a2) and (a3) are obvious. The construction also implies that

$$\frac{|L_i|}{4} \leq |L_{i+1}| \leq \frac{3|L_i|}{4} \text{ for } i \in [s-2] \text{ and } |L_{s-1}| = 1. \quad (2.1)$$

Note that we have $s \geq 2$ because $d \geq 1$. This implies

$$\left(\frac{4}{3}\right)^{s-i-1} \leq |L_i| \leq 4^{s-i-1} \text{ for } i \in [s-1]. \quad (2.2)$$

In particular, (2.2) with $i = 1$ and the fact that $d = |L_1|$ together imply

$$\frac{1}{2} \log(d) + 2 \leq s \leq \frac{\log(d)}{2 - \log(3)} + 2 \leq \frac{5}{2} \log(d) + 2.$$

Thus we get (a1).

Note that $L_i \setminus (L_{i+1} \cup \{v_{i+1}\}) \subseteq N_T^+(v_i) \setminus A$ and $L_{i-1} \setminus (L_i \cup \{v_i\}) \subseteq N_T^-(v_i)$. Thus, for $1 \leq s/5 - 12$ we have

$$|N_T^+(v_i) \setminus A| \geq |L_i \setminus L_{i+1}| - 1 \stackrel{(2.1)}{\geq} \frac{1}{4}|L_i| - 1 \stackrel{(2.2)}{\geq} \frac{1}{4}\left(\frac{4}{3}\right)^{s-i-1} - 1 \geq 8d^{1/7} - 1$$

Here, we get the final inequality because $s - i - 1 \geq 4s/5 + 11 \stackrel{(a1)}{\geq} \frac{2}{5} \log(d+1) + 12$. Similarly we also get $|N_T^-(v_i) \setminus A| \geq |L_{i-1} \setminus L_i| - 1 \geq 8d^{1/7} - 1$. Thus (a4) holds.

For $i \leq s - 5 \log(k) - 30$, (2.2) implies that

$$|L_i| \geq \left(\frac{4}{3}\right)^{s-i-1} > 4000k^2.$$

Hence

$$|N_T^+(v_i) \setminus A| \geq |L_i \setminus L_{i+1}| - 1 \stackrel{(2.1)}{\geq} \frac{1}{4}|L_i| - 1 \geq 1000k^2, \quad |N_T^+(v_i) \setminus A| \geq |L_{i-1} \setminus L_i| - 1 \geq 1000k^2.$$

Thus we get (a5). \square

By reversing arcs of a tournament T in Lemma 2.2, we have the following analogue.

Lemma 2.3. *Let v be a vertex in an n -vertex tournament T with $d = d_T^-(v)$. Then there exist $B \subseteq V(T)$ and a vertex $b \in B$ such that the following properties hold:*

- (b1) *We have $\frac{1}{2} \log(d+1) + 1 \leq s \leq \frac{5}{2} \log(d+1) + 2$ where $s = |B|$*
- (b2) *$T[B]$ is a transitive tournament with respect to the ordering (v_1, \dots, v_s) with source b and sink v .*
- (b3) *B out-dominates $V(T) \setminus B$.*
- (b4) *For $i \geq 4s/5 + 13$, we have*

$$|N_T^+(v_i) \setminus B|, |N_T^-(v_i) \setminus B| \geq 8d^{1/7} - 1.$$

- (b5) *For $i \geq 5 \log(k) + 31$, we have*

$$|N_T^+(v_i) \setminus B|, |N_T^-(v_i) \setminus B| \geq 1000k^2.$$

3. SPARSE LINKAGE STRUCTURE

In this section, we will prove Lemma 3.4. For an ordering $\sigma = (v_1, \dots, v_n)$ of vertices, we say that an arc $\overrightarrow{v_i v_j}$ is σ -forward if $i < j$, and σ -backward if $j < i$. For two integers a, b , we let $\sigma(a, b) := \{v_\ell : a \leq \ell \leq b, \ell \in [n]\}$. For positive integers n, k, t , an n -vertex digraph D and an ordering σ of $V(D)$, we say an D is (σ, k, t) -good if it satisfies the following.

- (D1) Every arc in D is a σ -forward arc.
- (D2) Every vertex in $\sigma(1, n-t)$ has out-degree at least k in D'
- (D3) Every vertex in $\sigma(t+1, n)$ has in-degree at least k in D' .

Note that if $n \leq t$, then $\sigma(1, n-t) = \sigma(t+1, n) = \emptyset$, so (D2) and (D3) are vacuous. Also note that (D2) or (D3) never holds together with (D1) if $t < k$. In Lemma 3.4, we will show that every tournament has a spanning subgraph D and an ordering σ such that D is a sparse (σ, k, t) -good digraph for appropriate k, t . The following shows that (σ, k, t) -good digraph D' provides a sparse linkage structure from/to certain vertex sets.

Claim 3.1. *Let k, t be two positive integers with $t \geq k$. Let D' be a (σ, k, t) -good digraph for an ordering σ of $V(D')$. Then for a set $S \subseteq V(D')$ of $k-1$ vertices and $v \in V(D') \setminus S$, there exists a path P in $D' - S$ from v to $\sigma(n-t+1, n)$ and a path P' in $D' - S$ from $\sigma(1, t)$ to v .*

Proof. If $n \leq t$, then the claim is trivial as $\sigma(n-t+1, n) = \sigma(1, t) = V(D')$. Assume $n \geq t+1$. Let $\sigma = (v_1, \dots, v_n)$. Take a path P starting at v and ending at v_j with the largest possible j . If $j \leq n-t$, then (D1) and (D2) imply that v_j has at least k out-neighbors with larger indices. Thus $N_{D'}(v_j) \setminus S$ contains a vertex $v_{j'}$ with $j' > j$. However, $P \cup \{\overrightarrow{v_j v_{j'}}\}$ contradicts the maximality of j . Thus we have $j > n-t$. Therefore there exists a path P in $T - S$ from v to $v_j \in \sigma(n-t+1, n)$. We can find P' in a similar way. \square

The following two claims are useful to prove Lemma 3.4.

Claim 3.2. For an integer $s \geq 0$, let G be a bipartite graph with bipartition $A \cup B$ with $A = \{a_1, \dots, a_n\}, B = \{b_1, \dots, b_n\}$ satisfying the following.

(P1_s) For all $i, j \in [n]$ with $i < j$, we have $|N_G(a_i) \cap \{b_{i+1}, \dots, b_j\}| \geq \frac{j-i-s}{2}$,

(P2_s) for all $i, j \in [n]$ with $i < j$, we have $|N_G(b_j) \cap \{a_i, \dots, a_{j-1}\}| \geq \frac{j-i-s}{2}$.

Then G contains a matching of size at least $n - s - 1$.

Proof. We may assume that $n - s - 1 > 0$, otherwise the claim is obvious. By König's theorem, it is enough to show that minimum vertex cover has size at least $n - s - 1$. Assume we have a vertex cover W of G . If $A \subseteq W$ or $B \subseteq W$, then $|W| \geq n \geq n - s - 1$. So we may assume that each of $A \setminus W$ and $B \setminus W$ contains an element. Consider the smallest index i such that $a_i \in A \setminus W$, and the largest index j such that $b_j \in B \setminus W$. We have $i < j$, otherwise W contains at least $n - 1$ vertices. Then we have

$$\{a_1, \dots, a_{i-1}\} \cup \{b_{j+1}, \dots, b_n\} \cup (N_G(b_j) \cap \{a_i, \dots, a_{j-1}\}) \cup (N_G(a_i) \cap \{b_{i+1}, \dots, b_j\}) \subseteq W.$$

By (P1_s) and (P2_s), we have

$$|W| \geq i - 1 + (n - j) + \frac{j - i - s}{2} + \frac{j - i - s}{2} \geq n - s - 1$$

as desired. \square

Claim 3.3. For $s \geq 0$, let D be an n -vertex oriented graph with $\delta^0(D) \geq n - s - 1$. Then there exists an ordering $\sigma = (v_1, \dots, v_n)$ of $V(D)$ that satisfies the following.

(Q1_s) For any $i, j \in [n]$ with $i < j$, v_i has at least $\frac{j-i-s}{2}$ out-neighbours in $\{v_{i+1}, \dots, v_j\}$,

(Q2_s) For any $i, j \in [n]$ with $i < j$, v_j has at least $\frac{j-i-s}{2}$ in-neighbours in $\{v_i, \dots, v_{j-1}\}$.

Moreover, we can find such an ordering in polynomial-time on n .

Proof. Note that such ordering exists because an ordering σ which maximizes the number of σ -forward arcs in D also satisfies both (Q1_s) and (Q2_s). Moreover, we can find a desired ordering in polynomial-time in the following way.

We start with an arbitrary ordering $\sigma_1 = (v_1, \dots, v_n)$. Assume we have an ordering σ_ℓ for some $\ell \geq 1$. If σ_ℓ satisfies (Q1_s) and (Q2_s), then we are done. Otherwise consider (i, j) with $i < j$ that does not satisfy (Q1_s) or (Q2_s). We define a new ordering

$$\sigma_{\ell+1} := \begin{cases} (v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_j, v_i, v_{j+1}, \dots, v_n) & \text{if } (i, j) \text{ does not satisfy (Q1}_s\text{),} \\ (v_1, \dots, v_{i-1}, v_j, v_i, \dots, v_{j-1}, v_{j+1}, \dots, v_n) & \text{if } (i, j) \text{ does not satisfy (Q2}_s\text{).} \end{cases}$$

Note that $\sigma_{\ell+1}$ has at least one more σ -forward arc than σ_ℓ . The number of σ -forward arcs in D is at most $\binom{n}{2}$, so the procedure must end before we have $\sigma_{\binom{n}{2}}$. Thus we obtain a desired ordering in polynomial-time in n . \square

Now we prove Lemma 3.4. It will be frequently used in the proof of Theorem 1.3.

Lemma 3.4. For integers $s \geq 0$ and $k \geq 1$, let D be an n -vertex oriented graph with $\delta^0(D) \geq n - 1 - s$. Then there exist an ordering σ of $V(D)$ and a $(\sigma, k, 2k + s - 1)$ -good spanning digraph D' of D with $|E(D')| \leq kn - k + sk$.

Proof. If $n < 2k + s$, then an arbitrary ordering σ of $V(D)$ with a digraph D' with no arcs is $(\sigma, k, 2k + s - 1)$ -good. Thus we may assume that $n \geq 2k + s$. By Claim 3.3, we can find an ordering $\sigma = (v_1, \dots, v_n)$ which satisfies condition (Q1_s) and (Q2_s) in Claim 3.3. We consider an auxiliary bipartite graph H_0 with a bipartition $A \cup B$, where $A = \{v_1, \dots, v_n\}$ and $B = \{v'_1, \dots, v'_n\}$, such that $v_i v'_j \in H_0$ if and only if $\vec{v_i v'_j}$ is a σ -forward arc of D . (i.e. $i < j$ and $\vec{v_i v'_j} \in E(D)$.)

Note that the condition (Q1_s) and (Q2_s) imply that the graph H_0 satisfies the condition (P1_s) and (P2_s). Assume we have a graph H_ℓ satisfying the condition (P1_{s+2ℓ}) and (P2_{s+2ℓ}).

By Claim 3.2, H_ℓ contains a matching M_ℓ of size at least $n - s - 2\ell - 1$. Let $H_{\ell+1} := H_\ell \setminus M_\ell$. Then for any $i, j \in [n]$, we have $|N_{H_\ell}(a_i) \setminus N_{H_{\ell+1}}(a_i)| \leq 1$ and $|N_{H_\ell}(b_j) \setminus N_{H_{\ell+1}}(b_j)| \leq 1$. Thus the graph $H_{\ell+1}$ satisfies the condition $(P1_{s+2\ell+2})$ and $(P2_{s+2\ell+2})$. Repeating this for $0 \leq \ell \leq k-1$ provides arc-disjoint matchings M_0, M_1, \dots, M_{k-1} of H_0 where the size of M_ℓ is at least $n - s - 2\ell - 1$ for $0 \leq \ell \leq k-1$. By deleting some arcs, we may assume that for $0 \leq \ell \leq k-1$ we have

$$|E(M_\ell)| = n - s - 2\ell - 1. \quad (3.1)$$

Let M be a subgraph of H_0 such that $E(M) := \bigcup_{\ell=0}^{k-1} E(M_\ell)$ and let D_1 be a subgraph of D such that

$$V(D_1) := V(D), \quad E(D_1) := \{\overrightarrow{v_i v_j} : v_i v_j \in E(M)\}.$$

Then by construction of H_0 , every arc of D_1 is a σ -forward arc and

$$\Delta(M) \leq k \quad \text{and} \quad |E(M)| = \sum_{\ell=0}^{k-1} |E(M_\ell)| \stackrel{(3.1)}{=} kn - k^2 - sk. \quad (3.2)$$

Also this implies that

$$\begin{aligned} \Delta^+(D_1) &\leq k, \quad \Delta^-(D_1) \leq k, \quad |E(D_1)| = kn - k^2 - sk, \\ d_{D_1}^-(v_i) &\leq \min\{k, i-1\} \quad \text{and} \quad d_{D_1}^+(v_i) \leq \min\{k, n-i\}. \end{aligned} \quad (3.3)$$

For each vertex $2k + s \leq i \leq n$, the number of σ -forward arcs towards v_i in T is at least $\lceil \frac{i-1-s}{2} \rceil \geq \lceil \frac{2k+s-1-s}{2} \rceil \geq k$ by $(P1_s)$. Thus for each $2k + s \leq i \leq n$, we can choose a set N_i^- of σ -forward arcs towards v_i such that $N_i^- \subseteq E(D) \setminus E(D_1)$ and $|N_i^-| = k - d_{D_1}^-(v_i)$. Similarly, for each $1 \leq i \leq 2k + s - 1$, we can choose a set N_i^+ of σ -forward arcs from v_i such that $N_i^+ \cap E(D_1) = \emptyset$ and $|N_i^+| = k - d_{D_1}^+(v_i)$. Define digraph $D' \subseteq D$ with

$$V(D') := V(D), \quad E(D') := E(D_1) \cup \bigcup_{i=2k+s}^n N_i^- \cup \bigcup_{i=1}^{n-2k-s+1} N_i^+.$$

Then D' satisfies (D1) by construction, and satisfies (D2) since $|d_{D'}^+(v_i)| \geq d_{D_1}^+(v_i) + |N_i^+| \geq k$ for $i \in [n - 2k - s + 2]$. Similarly, D' also satisfies (D3), thus D' is $(\sigma, k, 2k + s - 1)$ -good. Note that

$$\begin{aligned} \left| \bigcup_{i=2k+s}^n N_i^- \right| &\leq \sum_{i=2k+s}^n (k - d_{D_1}^-(v_i)) = k(n - 2k - s + 1) - \sum_{i=1}^n d_{D_1}^-(v_i) + \sum_{i=1}^{2k+s-1} d_{D_1}^-(v_i) \\ &\stackrel{(3.3)}{\leq} k(n - 2k - s + 1) - |E(D_1)| + \sum_{i=1}^{2k+s-1} \min\{k, i-1\} \stackrel{(3.3)}{=} \binom{k}{2} + sk. \end{aligned}$$

Here, we get the second inequality because $|E(D_1)| = \sum_{i=1}^n d_{D_1}^-(v_i)$. Similarly, we also have $|\bigcup_{i=1}^{n-2k-s+1} N_i^+| \leq \binom{k}{2} + sk$. Thus we have

$$\begin{aligned} |E(D')| &\leq |E(D_1)| + \left| \bigcup_{i=2k+s}^n N_i^- \right| + \left| \bigcup_{i=1}^{n-2k-s+1} N_i^+ \right| \\ &\stackrel{(3.3)}{\leq} kn - k^2 - sk + 2\binom{k}{2} + 2sk = kn - k + sk. \end{aligned}$$

□

4. SMALL TOURNAMENTS

In this section, we show that Theorem 1.3 holds for any strongly k -connected tournament T with at most $100k \log(k+1)$ vertices. Note that the Theorem 4.2 is sufficient for our purpose. To prove Theorem 4.2, we use the following lemma, which is a modification of Lemma 2.1 in [11], and the proof is almost identical except a few changes.

Lemma 4.1. [11] *Let $k \geq 1$ and $n \geq 5k$. Every n -vertex tournament T contains two disjoint sets of vertices X and Y of size k such that for any set S of $k-1$ vertices and any $x \in X \setminus S, y \in Y \setminus S$ there is a path P in $T - S$ from x to y .*

Proof. Let $\overrightarrow{K_{k,k}}$ be a bipartite digraph with partition A, B such that $|A| = |B| = k$ and for every $u \in A, v \in B$, we have $\overrightarrow{uv} \in E(\overrightarrow{K_{k,k}})$. If T contains $\overrightarrow{K_{k,k}}$ with bipartition A and B as a subgraph, then $X := A, Y := B$ are sufficient for our purpose. Thus we may assume that T does not contain $\overrightarrow{K_{k,k}}$ as a subgraph.

Let x_1, \dots, x_k be a set of k vertices in T of largest out-degree and y_1, \dots, y_k be a set of k vertices in T of largest in-degree. Let $X := \{x_1, \dots, x_k\}, Y := \{y_1, \dots, y_k\}$. From Lemma 2.1 $d_T^+(x_i) \geq (n-k)/2 \geq 2k$ and $d_T^-(y_i) \geq (n-k)/2 \geq 2k$ for $i \in [k]$. Consider a set $S \subseteq V(T)$ of size $k-1$. For each $i, j \in [k]$ let $X_{i,j} := N^+(x_i) \setminus N^-(y_j)$, $Y_{i,j} := N^-(y_j) \setminus N^+(x_i)$, $I_{i,j} = N^+(x_i) \cap N^-(y_j)$. Let $M_{i,j}$ be a maximum matching between $X_{i,j}$ and $Y_{i,j}$ such that every arc is directed from $X_{i,j}$ to $Y_{i,j}$. For each $z \in I_{i,j}$, T contains a path (x_i, z, y_j) and for each $\overrightarrow{ww'} \in M_{i,j}$, T contains a path (x_i, w, w', y_j) . Moreover, those paths are all pairwise internally vertex disjoint. Thus if $|M_{i,j}| + |I_{i,j}| \geq k$ for all $i, j \in [k]$, then for any x_i and y_j , there are at least k internally vertex disjoint paths from x_i to y_j . So we are done since for each $i, j \in [k]$ at least one path from x_i to y_j does not intersect with S . If there exist $i, j \in [k]$ such that $|M_{i,j}| + |I_{i,j}| < k$, then we have

$$|X_{i,j} \setminus V(M_{i,j})| \geq |N_T^+(x_i) - I_{i,j} - V(M_{i,j})| \geq d_T^+(x_i) - k \geq k.$$

Similarly we get $|Y_{i,j} \setminus V(M_{i,j})| \geq k$. Since $M_{i,j}$ is a maximal matching from $X_{i,j}$ to $Y_{i,j}$, for any $x' \in X_{i,j} \setminus V(M_{i,j})$ and $y' \in Y_{i,j} \setminus V(M_{i,j})$ we have $\overrightarrow{y'x'} \in E(T)$. It contradicts that T does not contain $\overrightarrow{K_{k,k}}$. \square

Theorem 4.2. *For integer $k \geq 1$, every strongly k -connected tournament T contains strongly k -connected spanning subgraph D with $|E(D)| \leq (5k-2)n + \binom{5k}{2}$.*

Proof. If T has less than $5k$ vertices, then T itself is sufficient to be D . Otherwise, let $V' \subseteq V$ be a set of $5k$ vertices. By applying Lemma 4.1, we can find two disjoint sets $X = \{x_1, \dots, x_k\}, Y = \{y_1, \dots, y_k\}$ of size k such that for any set $S \subseteq V'$ of size $k-1$ and vertices $x \in X, y \in Y$, there exists a path from x to y in $T[V'] - S$. We apply Lemma 3.4 to T with parameters $0, k$ corresponding to s, k , and we obtain an ordering $\sigma = (v_1, \dots, v_n)$ of $V(T)$ and a $(\sigma, k, 2k-1)$ -good spanning digraph $D' \subseteq T$ with $|E(D')| \leq kn - k$. For each $n-2k+2 \leq i \leq n$, we take $\{P(v_i, j) : j \in [k]\}$ a k -fan from v_i to X such that $P(v_i, j)$ is a path from v_i to x_j (such a path exists since T is strongly k -connected). For each $1 \leq i \leq 2k-1$, we take $\{Q(v_i, j) : j \in [k]\}$ a k -fan from Y to v_i such that $Q(v_i, j)$ is a path from y_j to v_i . Then for each $n-2k+2 \leq i \leq n$ and $1 \leq i' \leq 2k-1$

$$\sum_{j=1}^k |E(P(v_i, j))| \leq n-1, \quad \sum_{j=1}^k |E(Q(v_{i'}, j))| \leq n-1,$$

because no vertex other than v_i is covered by two distinct paths in a k -fan from v_i to X or by two distinct paths in a k -fan from Y to v_i . Let D be subgraph of T such that

$$V(D) := V(T), \quad E(D) := E(T(V')) \cup E(D') \cup \bigcup_{i=1}^{2k-1} \bigcup_{j=1}^k Q(v_i, j) \cup \bigcup_{i=n-2k+2}^n \bigcup_{j=1}^k P(v_i, j).$$

Then

$$\begin{aligned} |E(D)| &\leq |E(T(V'))| + |E(D')| + (2k-1)(n-1) + (2k-1)(n-1) \\ &\leq \binom{5k}{2} + kn - k + (4k-2)n \leq (5k-2)n + \binom{5k}{2}. \end{aligned}$$

Moreover, for any set $S \subseteq V(D)$ of $k-1$ vertices and any vertices $u, v \in V(T) \setminus S$, there are a path P from v to v_i and a path P' from $v_{i'}$ to u in $D' - S$ for some $i \geq n-2k+2$ and $i' \leq 2k+1$, by Claim 3.1. Since $\{P(v_i, j) : j \in [k]\}$ and $\{Q(v_{i'}, j) : j \in [k]\}$ are k -fans, there are $s, s' \in [k]$ such that both $P(v_i, s)$ and $Q(v_{i'}, s')$ do not intersect S . By Claim 4.1, there is a path P'' in $T[V'] - S$ from x_s to $y_{s'}$. Hence $E(P) \cup E(P(v_i, s)) \cup E(P'') \cup E(Q(v_{i'}, s')) \cup E(P')$ contains a path in $D - S$ from u to v . Thus D is strongly k -connected. \square

5. PROOF OF THEOREM 1.3

In this section, we prove Theorem 1.3. First we briefly sketch the idea. For given strongly k -connected tournament T , we construct a set A which is the union of many in-dominating sets and a set B which is the union of many out-dominating sets and k paths P_1, \dots, P_k from A to B . We divide the vertex set $V(T)$ into V_1, V'_1, V_2, V_3, V_4 and apply Lemma 3.4 to each set to find a sparse subgraph D and two sets W^+ and W^- such that D contains many paths from any vertex u to W^+ and many paths from W^- to any vertex v . We also add some arcs to the subgraph D so that there are many paths in D from each vertex in W^+ to A , and many paths in D from B to each vertex in W^- . Then these paths with paths P_1, \dots, P_k form many paths from u to v . It holds for any pair of vertices u and v , thus D has high connectivity while D is sparse.

Now we start the proof by considering a strongly k -connected n -vertex tournament T . Let $V := V(T)$. Note that the Theorem is trivial if $k = 1$ because every strongly connected n -vertex tournament contains a cycle of length l for all $3 \leq l \leq n$, hence a Hamilton cycle (See [2] or [10]). Furthermore, the proof is constructive, and provides a polynomial-time algorithm to find a Hamilton cycle in a strongly connected tournament.

If $n \leq 100k \log(k+1)$, then Theorem 4.2 implies Theorem 1.3. Thus we may assume the following.

$$k \geq 2, \quad n > 100k \log(k+1).$$

Now we construct appropriate in-dominating set A and out-dominating set B as we sketched before. Let X and Y be two disjoint sets such that X is a set of $3k-1$ vertices with smallest out-degrees, and let Y is a set of $3k-1$ vertices with smallest in-degrees. Let $\delta^- := \max_{y \in Y} d_T^-(y)$ and $\delta^+ := \max_{x \in X} d_T^+(x)$. Without loss of generality, we assume

$$\delta^- \geq \delta^+. \quad (5.1)$$

Choose $x_1 \in X$ having the largest number of out-neighbors in $V \setminus (X \cup Y)$ among all vertices in X , and let

$$d_1^+ := |(V \setminus (X \cup Y)) \cap N_T^+(x_1)|.$$

We apply Lemma 2.2 with $T - (X \cup Y), x_1, d_1^+$ corresponding to $T - \{v\}, v, d$ to find a set A_1 and a sink vertex $a_1 \in A_1$ satisfying (a1)–(a5). For given x_1, \dots, x_i and A_1, \dots, A_i , choose $x_{i+1} \in X \setminus \{x_1, \dots, x_i\}$ having the largest number of out-neighbours in $V \setminus (X \cup Y \cup \bigcup_{j=1}^i A_j)$ among all the vertices in $X \setminus \{x_1, \dots, x_i\}$ and let

$$d_{i+1}^+ := |(V \setminus (X \cup Y \cup \bigcup_{j=1}^i A_j)) \cap N_T^+(x_{i+1})|.$$

We apply Lemma 2.2 with $T - (X \cup Y \cup \bigcup_{j=1}^i A_j), x_{i+1}, d_{i+1}^+$ corresponding to $T - \{v\}, v, d$ to find a set A_{i+1} and a sink vertex $a_{i+1} \in A_{i+1}$ satisfying (a1)–(a5). By repeating this $3k-1$ times, we get A_1, \dots, A_{3k-1} and a_1, \dots, a_{3k-1} . We let $A := \bigcup_{i=1}^{3k-1} A_i$.

Next, we choose $y_1 \in Y$ having the largest number of in-neighbours in $V \setminus (X \cup Y \cup A)$. Let

$$d_1^- := |(V \setminus (X \cup Y \cup A)) \cap N_T^-(y_1)|.$$

Then we apply Lemma 2.3 with $T - (X \cup Y \cup A)$, y_1, d_1^- corresponding to $T - \{y\}, v, d$ to find a set B_1 and a source vertex $b_1 \in B_1$ satisfying (b1)–(b5). For given A, y_1, \dots, y_i and B_1, \dots, B_i , choose $y_{i+1} \in Y \setminus \{y_1, \dots, y_i\}$ having the largest number of in-neighbours in $V \setminus (X \cup Y \cup A \cup \bigcup_{j=1}^i B_j)$ among all the vertices in $Y \setminus \{y_1, \dots, y_i\}$ and let

$$d_{i+1}^- := |(V \setminus (X \cup Y \cup A \cup \bigcup_{j=1}^i B_j)) \cap N_T^-(y_{i+1})|.$$

We apply Lemma 2.3 with $T - (X \cup Y \cup A \cup \bigcup_{j=1}^i B_j)$, y_{i+1}, d_{i+1}^- corresponding to $T - \{v\}, v, d$ to find a set B_{i+1} and a source vertex $b_{i+1} \in B_{i+1}$ satisfying (b1)–(b5). By repeating this $3k-1$ times, we get B_1, \dots, B_{3k-1} and b_1, \dots, b_{3k-1} . We let $B := \bigcup_{i=1}^{3k-1} B_i$. Note that $T[B_i]$ is a transitive tournament for each $i \in [3k-1]$. For each i , we let B'_i be the set of last $\lceil |B_i|/5 - 12 \rceil$ vertices, and let B''_i be the set of first $\lceil 5 \log(k) + 30 \rceil$ vertices in the transitive ordering of $T[B_i]$, respectively. We define

$$A_{\text{sink}} := \{a_1, \dots, a_{3k-1}\}, B_{\text{source}} := \{b_1, \dots, b_{3k-1}\}, B' := \bigcup_{i=1}^{3k-1} B'_i, \text{ and } B'' := \bigcup_{i=1}^{3k-1} B''_i.$$

From this construction, we get numbers $d_1^+, \dots, d_{3k-1}^+, d_1^-, \dots, d_{3k-1}^-$ satisfying

$$\delta^+ \geq d_1^+ \geq d_2^+ \geq \dots \geq d_{3k-1}^+ \quad \text{and} \quad \delta^- \geq d_1^- \geq d_2^- \geq \dots \geq d_{3k-1}^-, \quad (5.2)$$

and sets $A_1, \dots, A_{3k-1}, B_1, \dots, B_{3k-1}, B'_1, \dots, B'_{3k-1}, B''_1, \dots, B''_{3k-1}$ and vertices $a_1, \dots, a_{3k-1}, b_1, \dots, b_{3k-1}$ satisfying the following (A1)–(A3) and (B1)–(B6) for all $i \in [3k-1]$.

- (A1) $\frac{1}{2} \log(d_i^+ + 1) + 1 \leq |A_i| \leq \frac{5}{2} \log(d_i^+ + 1) + 2$,
- (A2) $T[A_i]$ is a transitive tournament with source x_i and sink a_i ,
- (A3) A_i in-dominates $V \setminus (A \cup B)$,
- (B1) $\frac{1}{2} \log(d_i^- + 1) + 1 \leq |B_i| \leq \frac{5}{2} \log(d_i^- + 1) + 2$,
- (B2) $T[B_i]$ is a transitive tournament with sink y_i and source b_i ,
- (B3) B_i out-dominates $V \setminus (A \cup B)$,
- (B4) $|B'_i| \geq |B_i|/5 - 12$ and for $v \in B'_i$ we have

$$|N_T^+(v) \setminus (A \cup \bigcup_{j=1}^i B_j)| \geq 8(d_i^-)^{1/7} - 1, \quad |N_T^-(v) \setminus (A \cup \bigcup_{j=1}^i B_j)| \geq 8(d_i^-)^{1/7} - 1,$$

- (B5) $|B''_i| \leq 5 \log(k) + 30$ and for $v \in B_i \setminus B''_i$ we have

$$|N_T^+(v) \setminus (A \cup \bigcup_{j=1}^i B_j)| \geq 1000k^2, \quad |N_T^-(v) \setminus (A \cup \bigcup_{j=1}^i B_j)| \geq 1000k^2.$$

- (B6) For a vertex $v \in B_i \setminus B'_i$, is satisfies $B'_i \subseteq N_T^+(v)$.

By Lemma 2.1, each of $T[A_{\text{sink}}]$ and $T[B_{\text{source}}]$ contains k vertices of in-degree at least k and k vertices of out-degree at least k . Let a_{i_1}, \dots, a_{i_k} be k distinct vertices having in-degree at least k in $T[A_{\text{sink}}]$ and let b_{j_1}, \dots, b_{j_k} be distinct k vertices having out-degree at least k in $T[B_{\text{source}}]$. By (A1), (B1) and the fact that $\delta^- \leq n-1$, we have $|A \cup B| \leq (6k-2)(\frac{5}{2} \log(n) + 2) < n-k$ since $n \geq 100k \log(k+1)$. Thus we have

$$|V \setminus (A \cup B)| \geq k. \quad (5.3)$$

Before starting the construction of the desired directed graph D , we prove Claim 5.1 and Claim 5.3 showing that for any $v \in A \cup B$ there exist a k -fan from v to $V \setminus (A \cup B)$ and a k -fan from $V \setminus (A \cup B)$ to v consisting of short paths.

Claim 5.1. For any vertex $v \in A \cup B$, we can find a k -fan $\{P^-(v, 1), \dots, P^-(v, k)\}$ from $V \setminus (A \cup B)$ to v such that $\sum_{i=1}^k |E(P^-(v, i))| \leq 70k \log(k+1)$.

To prove Claim 5.1, note that (5.1), (5.2), (A1) and (B2) together imply that

$$|A \cup B| \leq (6k-2)(2.5 \log(\delta_1^- + 1) + 2). \quad (5.4)$$

We consider the following two cases.

Case 1. $\delta_1^- \leq 60k^2$.

In this case, consider $\{P^-(v, 1), \dots, P^-(v, k)\}$, a k -fan from $V \setminus (A \cup B)$ to v . Such a k -fan exists because of Fact 1 and (5.3). By (5.4), we have $|A \cup B| \leq (6k-2)(2.5 \log(60k^2 + 1) + 2) \leq 69k \log(k+1)$. Since every vertex in each $P^-(v, i)$ is in $A \cup B$ except for one vertex, we have $\sum_{i=1}^k |E(P^-(v, i))| \leq |A \cup B| + k \leq 70k \log(k+1)$.

Case 2. $\delta_1^- > 60k^2$.

Since $k \geq 2$, we have

$$\delta_1^- \geq (6k-2)(2.5 \log(\delta_1^- + 1) + 2) + 2k \stackrel{(5.4)}{\geq} |A \cup B| + 2k.$$

Thus for any vertex $u \notin Y$, we have $d^-(u) \geq \delta_1^- \geq |A \cup B| + 2k$.

If $v \notin Y$, take k distinct paths of length 1 from $V \setminus (A \cup B)$ to v , and let $P^-(v, 1), \dots, P^-(v, k)$ be those paths of length 1. Then we have $\sum_{i=1}^k |E(P^-(v, i))| \leq k \leq 70k \log(k+1)$. If $v \in Y$, then take $\{Q_1, \dots, Q_k\}$, a k -fan from $V \setminus Y$ to v given by Fact 1 and (5.3). Let v_i be the starting vertex of Q_i for $1 \leq i \leq k$. Then we have

$$\sum_{i=1}^k |E(Q_i)| \leq |Y| + k \leq 4k - 1.$$

Consider $i \in [k]$ with $v_i \in A \cup B$. Since each v_i is not in Y , it has at least $2k$ in-neighbors outside $A \cup B$. For each $i \in [k]$ with $v_i \in A \cup B$, we choose v'_i in $N_T^-(v_i) \cap (V(T) \setminus (A \cup B \cup \{v_1, \dots, v_k\}))$ in the way that v'_i 's are all distinct. Let

$$P^-(v, i) := \begin{cases} Q_i \cup \{\overrightarrow{v'_i v_i}\} & \text{if } v_i \in A \cup B, \\ Q_i & \text{if } v_i \notin A \cup B. \end{cases}$$

Then $\{P^-(v, 1), \dots, P^-(v, k)\}$ form a k -fan from $V \setminus (A \cup B)$ to v such that

$$\sum_{i=1}^k |E(P^-(v, i))| \leq k + \sum_{i=1}^k |E(Q_i)| \leq |Y| + 2k = 5k - 1 \leq 70k \log(k+1).$$

This proves Claim 5.1.

Claim 5.2. For each $v \in A \cup B''$, there exists a k -fan $\{P_*^+(v, 1), \dots, P_*^+(v, k)\}$ from v to $V \setminus (A \cup B'')$ such that $\sum_{i=1}^k |E(P_*^+(v, i))| \leq 98k \log(k+1)$.

Note that we have

$$\begin{aligned} |A \cup B''| &\stackrel{(A1)}{\leq} \sum_{i=1}^{3k-1} \left(\frac{5}{2} \log(d_i^+ + 1) + 2 \right) + |B''| \\ &\stackrel{(5.2), (B5)}{\leq} (3k-1) \left(\frac{5}{2} \log(\delta^+ + 1) + 2 \right) + (3k-1)(5 \log(k) + 30) \end{aligned} \quad (5.5)$$

To prove Claim 5.2, we consider the following two cases.

Case 1. $\delta^+ \leq 100k^2$.

Since T is strongly k -connected, there exists $\{P_*^+(v, 1), \dots, P_*^+(v, k)\}$, a k -fan from v to $V \setminus (A \cup B'')$ by Fact 1 and (5.3). Since $P_*^+(v, 1), \dots, P_*^+(v, k)$ contains at most k vertices outside $A \cup B''$ and $\delta^+ \leq 100k^2$, we have

$$\sum_{i=1}^{3k-1} |E(P_*^+(v, i))| \leq |A \cup B''| + k \stackrel{(5.5)}{\leq} 98k \log(k+1).$$

Case 2. $\delta^+ \geq 100k^2$.

In this case, we have

$$|A \cup B''| + 2k \stackrel{(5.5)}{\leq} (3k-1)\left(\frac{5}{2}\log(\delta^+ + 1) + 2\right) + (3k-1)(5k \log(k) + 30) + 2k \leq \delta^+$$

If $v \notin X$, then $d_T^+(v) \geq \delta^+ \geq |A \cup B''| + 2k$. So we can find k paths Q'_1, \dots, Q'_k of length 1 from v to $V \setminus (A \cup B'')$. Let $P_*^+(v, 1), \dots, P_*^+(v, k)$ be those paths of length 1. Then $\sum_{i=1}^k |E(P_*^+(v, i))| \leq k \leq 98k \log(k+1)$.

If $v \in X$, then we find a k -fan $\{Q'_1, \dots, Q'_k\}$ from v to $V \setminus X$ by Fact 1 and (5.3). Then because all vertices of Q'_i except the last vertex belong to X , we have $\sum_{i=1}^k |E(Q'_i)| \leq |X| + k$. Let u'_i be the end vertex of Q'_i , for $1 \leq i \leq k$. Consider $i \in [k]$ with $u'_i \in A \cup B''$. Since $u'_i \notin X$ and has at least $2k$ out-neighbors in $V \setminus (A \cup B'')$, we can choose $u''_i \in N_T(u'_i) \setminus (A \cup B'' \cup \{u'_1, \dots, u'_k\})$ such that u''_i 's are distinct. We let

$$P_*^+(v, i) := \begin{cases} Q'_i \cup \{\overrightarrow{u'_i u''_i}\} & \text{if } u'_i \in A \cup B'', \\ Q'_i & \text{if } u'_i \notin A \cup B''. \end{cases}$$

Then we have k -fan $\{P_*^+(v, 1), \dots, P_*^+(v, k)\}$ from v to $V \setminus (A \cup B'')$ such that

$$\sum_{i=1}^k |E(P_*^+(v, i))| \leq \sum_{i=1}^k |E(Q'_i)| + k \leq |X| + 2k = 5k - 1 \leq 98k \log(k+1).$$

This proves Claim 5.2.

Now we prove Claim 5.3 by using Claim 5.2.

Claim 5.3. *For any vertex $v \in A \cup B$, there exists a k -fan $\{P^+(v, 1), \dots, P^+(v, k)\}$ from v to $V \setminus (A \cup B)$ with $\sum_{i=1}^k |E(P^+(v, i))| \leq 100k \log(k+1)$.*

To prove Claim 5.3, we first use Claim 5.2 to find a k -fan from v to $V \setminus (A \cup B'')$ such that $\sum_{i=1}^k |E(P_*^+(v, i))| \leq 98k \log(k+1)$. Let u_i be the last vertex in $P_*^+(v, i)$ and let $U := \{u_1, \dots, u_k\}$. Then for each $i \in [k]$ all vertices in $P_*^+(v, i)$ except u_i belong to $A \cup B''$, and u_i is either in $V \setminus (A \cup B)$ or in $B \setminus B''$. For each i with $u_i \in B \setminus B''$, let ℓ_i be the index such that $u_i \in B_{\ell_i}$. Then we can partition $[k]$ into four sets I_1, I_2, I_3 and I_4 as follows.

- For $i \in I_1$, we have $|B_{\ell_i}| \geq 18k + 80, u_i \notin B'_{\ell_i}$,
- for $i \in I_2$, we have $|B_{\ell_i}| \geq 18k + 80, u_i \in B'_{\ell_i}$
- for $i \in I_3$, we have $|B_{\ell_i}| < 18k + 80$,
- for $i \in I_4$, we have $u_i \notin A \cup B$.

First, consider $i \in I_1 \cup I_2$. Since $|B_{\ell_i}| \geq 18k + 80$, (B1) implies that

$$d_{\ell_i}^- \geq 2^{\frac{1}{2.5}(|B_{\ell_i}|-2)} - 1 \geq 2^{7k+30}, \quad (5.6)$$

and (B4) implies that

$$|B'_{\ell_i}| \geq 3k. \quad (5.7)$$

For any $u \in B'_{\ell_i}$ we have

$$\begin{aligned}
|N_T^+(u) \setminus (A \cup B)| &\geq \left| N_T^+(u) \setminus (A \cup \bigcup_{p=1}^{\ell_i} B_p) \right| - \left| \bigcup_{p=\ell_i+1}^{3k-1} B_p \right| \\
&\stackrel{(B4)}{\geq} 8(d_{\ell_i}^-)^{1/7} - 1 - \left| \bigcup_{p=\ell_i+1}^{3k-1} B_p \right| \\
&\stackrel{(5.6)}{\geq} (3k-1)(2.5 \log(d_{\ell_i}^- + 1) + 2) + 3k - \left| \bigcup_{p=\ell_i+1}^{3k-1} B_p \right| \\
&\stackrel{(B1), (5.2)}{\geq} 3k. \tag{5.8}
\end{aligned}$$

Here, we get the third inequality since $8x^{1/7} - 1 \geq (3k-1)(2.5 \log(x+1) + 2) + 3k$ holds for $x \geq 2^{7k+30}$ and $k \geq 2$. Thus any vertex $u \in B'_{\ell_i}$ has at least $3k$ out-neighbors in $V \setminus (A \cup B)$.

For $i \in I_1$, (5.7) implies that $|B'_{\ell_i}| \geq 3k$ and (B6) implies that $B'_{\ell_i} \subseteq N_T^+(u_i)$. From this we obtain $|(N_T^+(u_i) \cap B'_{\ell_i}) \setminus U| = |B'_{\ell_i} \setminus U| \geq 3k - k \geq 2k$. Thus we can choose a set $W = \{w_i : i \in I_1\}$ of $|I_1|$ distinct vertices such that $w_i \in N_T^+(u_i) \cap (B'_{\ell_i} \setminus U)$. Again, (5.8) implies that

$$|N_T^+(w_i) \setminus (A \cup B \cup U \cup W)| \geq k,$$

so we can further choose a set $W' = \{w'_i : i \in I_1\}$ of $|I_1|$ distinct vertices such that $w'_i \in N_T^+(w_i) \setminus (A \cup B \cup U \cup W)$.

Now we consider $i \in I_2$. In this case $u_i \in B'_{\ell_i}$ and (5.8) imply that

$$|N_T^+(u_i) \setminus (A \cup B \cup U \cup W \cup W')| \geq 2k - 2|I_1| \geq |I_2|,$$

so we can further choose a set $W^* = \{w_i^* : i \in I_2\}$ of $|I_2|$ distinct vertices such that $w_i^* \in N_T^+(u_i) \setminus (A \cup B \cup U \cup W \cup W')$.

Now we consider $i \in I_3$. In this case, u_i belongs to $B_{\ell_i} \setminus B''_{\ell_i}$. Thus

$$\begin{aligned}
|N_T^+(u'_i) \setminus (A \cup B)| &\geq \left| N_T^+(u'_i) \setminus (A \cup \bigcup_{p=1}^{\ell_i} B_p) \right| - \left| \bigcup_{p=\ell_i+1}^{3k-1} B_p \right| \\
&\stackrel{(B1), (B5)}{\geq} 1000k^2 - \sum_{p=\ell_i+1}^{3k-1} (2.5 \log(d_p^- + 1) + 2) \\
&\stackrel{(5.2)}{\geq} 1000k^2 - (3k-1)(2.5 \log(d_{\ell_i}^- + 1) + 2) \\
&\stackrel{(B1)}{\geq} 1000k^2 - 5(3k-1)|B_{\ell_i}| \\
&\geq 1000k^2 - 5(3k-1)(18k+80) \geq 5k \geq |I_3| + 4k.
\end{aligned}$$

Thus we can choose a set $W^{**} := \{w_i^{**} : i \in I_3\}$ of $|I_3|$ distinct vertices such that $w_i^{**} \in N_T^+(u_i) \setminus (A \cup B \cup U \cup W \cup W' \cup W^*)$. Note that U, W, W', W^*, W^{**} are pairwise disjoint sets by construction. For $i \in [k]$, let $P^+(v, i)$ be a path from v to $V \setminus (A \cup B)$ as follows.

$$E(P^+(v, i)) := \begin{cases} E(P_*^+(v, i)) \cup \{\overrightarrow{u_i w_i}, \overrightarrow{w_i w_i'}\} & \text{if } i \in I_1, \\ E(P_*^+(v, i)) \cup \{\overrightarrow{u_i w_i^*}\} & \text{if } i \in I_2, \\ E(P_*^+(v, i)) \cup \{\overrightarrow{u_i w_i^{**}}\} & \text{if } i \in I_3, \\ E(P_*^+(v, i)) & \text{if } i \in I_4. \end{cases}$$

We claim that $\{P^+(v, i)\}_{i=1}^k$ is a k -fan from v to $V \setminus (A \cup B)$, and the sum of lengths is small. Indeed, for any $i \in [k]$, $P^+(v, i)$ is a path from v to $V \setminus (A \cup B)$.

paths $\{V(P^+(v, i)) \setminus \{v\}\}_{i=1}^k$ are pairwise disjoint because the paths $\{V(P_*^+(v, i)) \setminus \{v\}\}_{i=1}^k$ are pairwise-disjoint, and U, W, W', W^*, W^{**} are pairwise disjoint. Moreover,

$$\sum_{i=1}^k |E(P^+(v, i))| = \sum_{i=1}^k |E(P_*^+(v, i))| + 2|I_1| + |I_2| + |I_3| \leq 98k \log(k+1) + 2k \leq 100k \log(k+1).$$

This proves Claim 5.3.

Now we will find collections of arcs E_0, E_1, E_2, E_3, E_4 and E_5 which together construct a desired digraph D . Recall that the tournament T is strongly k -connected and $i_1, \dots, i_k, j_1, \dots, j_k$ are defined before Claim 5.1. By Menger's theorem, we can take k vertex-disjoint paths P_1, \dots, P_k from $\{a_{i_1}, \dots, a_{i_k}\}$ to $\{b_{j_1}, \dots, b_{j_k}\}$ so that each path is minimal. Because of the minimality, each path is backwards-transitive. By permuting indices, we may assume that P_s is a backwards-transitive path from a_{i_s} to b_{j_s} . Let $V^{\text{int}}(P_s)$ be the set of internal vertices of P_s . We define

$$V_1 := (A \cup B) \setminus \left(\bigcup_{i=1}^k V^{\text{int}}(P_i)\right), \quad V'_1 := (A \cup B) \cap \left(\bigcup_{i=1}^k V^{\text{int}}(P_i)\right). \quad (5.9)$$

Note that $\{a_{i_1}, \dots, a_{i_k}, b_{j_1}, \dots, b_{j_k}\} \subseteq V_1$. Let $E_0 := \bigcup_{s=1}^k E(P_s)$. Now we will find a set of arcs E_1 as in the following claim.

Claim 5.4. *There exist a set of arcs $E_1 \subseteq E(T)$ and a set of vertices $V_2 \subseteq V \setminus (A \cup B)$ satisfying the following.*

- (E1)₁ $|E_1| \leq k|V_1| + (k-1)|V'_1| + 680k^2 \log(k+1)$ and $|V_2| \leq 8k^2$.
- (E1)₂ For any set $S \subseteq V(T)$ of size $k-1$ and a vertex $v \in (V_1 \cup V'_1) \setminus S$, we can find a path P in $T - S$ from v to V_2 such that $E(P) \subseteq E_0 \cup E_1$.
- (E1)₃ For any set $S \subseteq V(T)$ of size $k-1$ and a vertex $v \in (V_1 \cup V'_1) \setminus S$, we can find a path P in $T - S$ from V_2 to v such that $E(P) \subseteq E_0 \cup E_1$.

To prove Claim 5.4, we apply Lemma 3.4 to $T[V_1]$ with parameters $0, k$ corresponding to s, k , respectively. Then we obtain an ordering σ_1 of V_1 with a $(\sigma_1, k, 2k-1)$ -good digraph $D_1 \subseteq T[V_1]$ such that $|E(D_1)| \leq k|V_1| - k$. We also consider a digraph $T[V'_1] - E_0$. Since $\delta^0(T[V'_1] - E_0) \geq |V'_1| - 3$, we can apply Lemma 3.4 to $T[V'_1] - E_0$ with parameters $2, (k-1)$ corresponding to s, k , respectively. Then we obtain an ordering σ'_1 of V'_1 and a $(\sigma'_1, k-1, 2k-1)$ -good digraph $D'_1 \subseteq T[V'_1] - E_0$ with $|E(D'_1)| \leq (k-1)|V'_1| + (k-1)$. Now we define W_1^- and W_1^+ as follows.

$$W_1^- := \sigma_1(1, 2k-1) \cup \sigma'_1(1, 2k-1) \text{ and } W_1^+ := \sigma_1(|V_1| - 2k + 1, |V_1|) \cup \sigma'_1(|V'_1| - 2k + 1, |V'_1|)$$

This gives

$$|W_1^-|, |W_1^+| \leq 4k - 2. \quad (5.10)$$

For each vertex $u \in W_1^-$ we use Claim 5.1 to obtain k almost-disjoint paths $P^-(u, 1), \dots, P^-(u, k)$ in T from $V \setminus (A \cup B)$ to u with

$$\sum_{i=1}^k |E(P^-(u, i))| \leq 70k \log(k+1). \quad (5.11)$$

For each vertex $u \in W_1^+$, we use Claim 5.3 to obtain k almost-disjoint paths $P^+(u, 1), \dots, P^+(u, k)$ in T from u to $V \setminus (A \cup B)$ with

$$\sum_{i=1}^k |E(P^+(u, i))| \leq 100k \log(k+1). \quad (5.12)$$

Let

$$E_1 := E(D_1) \cup E(D'_1) \cup \bigcup_{u \in W_1^-, i \in [k]} E(P^-(u, i)) \cup \bigcup_{u \in W_1^+, i \in [k]} E(P^+(u, i)), \quad (5.13)$$

$$V_2 := V(E_1) \setminus (V_1 \cup V'_1).$$

Since $V_1 \cup V'_1 = A \cup B$, any vertex in V_2 is either the last vertices of $P^+(u, i)$ for some $i \in [k]$ and $u \in W_1^+$ or the first vertex of $P^-(u, i)$ for some $i \in [k]$ and $u \in W_1^-$. Thus we have

$$|V_2| \leq k(|W_1^+| + |W_1^-|) \stackrel{(5.10)}{\leq} 8k^2. \text{ Moreover,}$$

$$\begin{aligned} |E_1| &\stackrel{(5.11), (5.12)}{\leq} |E(D_1)| + |E(D_2)| + 70k \log(k+1)|W_1^-| + 100k \log(k+1)|W_1^+| \\ &\stackrel{(5.10)}{\leq} k|V_1| + (k-1)|V_1| + 680k^2 \log(k+1). \end{aligned}$$

This shows (E1)₁. To show (E1)₂, let S be a set of $k-1$ vertices in V and let v be a vertex with $v \in (V_1 \cup V'_1) \setminus S$. We consider the following two cases.

Case 1. $v \in V_1$.

By Claim 3.1 and the fact that D_1 is $(\sigma_1, k, 2k-1)$ -good, we can find a path P' from v to a vertex $u \in W_1^+$ in $T-S$ such that $E(P') \subseteq E_1$. Also $P^+(u, 1), \dots, P^+(u, k)$ are disjoint paths except the common starting vertex $u \notin S$, thus there exists $j \in [k]$ such that $P^+(u, j)$ does not intersect with S . Then $E(P') \cup E(P^+(u, j))$ contains a path P in $T-S$ from v to V_2 with $E(P) \subseteq E_1$.

Case 2. $v \in V'_1$.

Assume $\sigma'_1 = (v'_1, \dots, v'_{|V'_1|})$. We consider a path P' in $D'_1 - S$ from v to v'_i with maximum i . If $i \geq |V'_1| - 2k + 2$, then we have $v'_i \in W_1^+$ and we can choose $j \in [k]$ such that $P^+(v'_i, j)$ does not intersect with S . Then $E(P') \cup E(P^+(v'_i, j))$ contains a path P in $T-S$ from v to V_2 with $E(P) \subseteq E_1$. If $i < |V'_1| - 2k + 2$, then the maximality of i implies $N_{D'_1}^+(v'_i) \subseteq S$ by (D1) and the fact that D'_1 is $(\sigma'_1, k-1, 2k-1)$ -good. Since

$$k-1 \stackrel{(D2)}{\leq} |N_{D'_1}^+(v'_i)| \leq |S| = k-1,$$

we have

$$S = N_{D'_1}^+(v'_i). \quad (5.14)$$

By (5.9) and the fact that $v'_i \in V'_1$, there exists $s \in [k]$ such that $v'_i \in V^{\text{int}}(P_s)$. We let P'' be the sub-path of P_s from v'_i to b_{j_s} . Since P_s is backwards-transitive, every vertex in $V(P'')$ belongs to $N_T^-(v'_i)$ except the first vertex v'_i and the second vertex, say u' , of P'' . Since $\overrightarrow{v'_i u'} \in E(P_s) \subseteq E_0$ and $D'_1 \subseteq T[V'_1] - E_0$, we obtain $\overrightarrow{v'_i u'} \notin E(D'_1)$. Thus $u' \notin N_{D'_1}^+(v'_i)$. This with the fact that $V(P'') \subseteq N_T^-(v'_i) \cup \{v'_i, u'\}$ implies that

$$V(P'') \cap S \subseteq (N_T^-(v'_i) \cup \{u'\}) \cap S \stackrel{(5.14)}{=} (N_T^-(v'_i) \cup \{u'\}) \cap N_{D'_1}^+(v'_i) = \emptyset.$$

Thus P'' does not intersect with S . Since $b_{j_s} \in V_1$, Case 1 implies that there exists a path P^* from b_{j_s} to V_2 in $T[V \setminus S]$ with $E(P^*) \subseteq E_1$. Then $E(P') \cup E(P'') \cup E(P^*)$ contains a path P in $T-S$ from v to V_2 with $E(P) \subseteq E_0 \cup E_1$. Thus we have (E1)₂. We can show (E1)₃ in a similar way. This proves Claim 5.4.

Claim 5.5. *There exist a set of arcs $E_2 \subseteq E(T)$ and two sets $W_2^+, W_2^- \subseteq V_2$ satisfying the following.*

$$(E2)_1 \quad |E_2| \leq k|V_2| - k \text{ and } |W_2^+|, |W_2^-| \leq 2k-1.$$

- (E2)₂ For a set $S \subseteq V(T)$ of size $k-1$ and a vertex $v \in V_2 \setminus S$, there exists a path P in $T-S$ from v to W_2^+ with $E(P) \subseteq E_2$.
- (E2)₃ For a set $S \subseteq V(T)$ of size $k-1$ and a vertex $v \in V_2 \setminus S$, there exists a path P in $T-S$ from W_2^- to v with $E(P) \subseteq E_2$.

To prove Claim 5.5, we apply Lemma 3.4 to $T[V_2]$ with parameters $0, k$ corresponding to s, k , respectively. Then we obtain an ordering σ_2 of V_2 and $(\sigma_2, k, 2k-1)$ -good digraph $D_2 \subseteq T[V_2]$ such that $|E(D_2)| \leq k|V_2| - k$. Let

$$E_2 := E(D_2), \quad W_2^- := \sigma_1(1, 2k-1) \quad \text{and} \quad W_2^+ := \sigma_1(|V_1| - 2k + 2, |V_1|),$$

then we have $|E_2| = |E(D_2)| \leq k|V_2| - k$ and $|W_2^-|, |W_2^+| \leq 2k-1$. Hence we have (E2)₁. By Claim 3.1 and the fact that D_2 is $(\sigma_2, k, 2k-1)$ -good, for any set S of $k-1$ vertices in V and a vertex $v \in V_2 \setminus S$, we can find a path P in $T-S$ from v to W_2^+ and a path P' in $T-S$ from W_2^- to v such that $E(P) \subseteq E_2, E(P') \subseteq E_2$. Thus we have the property (E2)₂ and (E2)₃. This proves Claim 5.5.

Now we define V_3, V_4 as follows.

$$V_3 := \bigcup_{i=1}^k V^{\text{int}}(P_i) \setminus (V_1' \cup V_2) \quad \text{and} \quad V_4 := V \setminus (V_1 \cup V_1' \cup V_2 \cup V_3). \quad (5.15)$$

Claim 5.6. *There exist a set of arcs $E_3 \subseteq E(T)$ disjoint from E_0 and two sets $W_3^+, W_3^- \subseteq V_3$ satisfying the following.*

- (E3)₁ $|E_3| \leq (k-1)|V_3| + (k-1)$ and $|W_3^+|, |W_3^-| \leq 2k-1$.
- (E3)₂ For a set $S \subseteq V(T)$ of size $k-1$ and a vertex $v \in V_3 \setminus S$, there exists a path P in $T-S$ from v to $W_3^+ \cup V_1$ with $E(P) \subseteq E_0 \cup E_3$.
- (E3)₃ For a set $S \subseteq V(T)$ of size $k-1$ and a vertex $v \in V_3 \setminus S$, there exists a path P in $T-S$ from $W_3^- \cup V_1$ to v with $E(P) \subseteq E_0 \cup E_3$.

To prove Claim 5.6, consider a digraph $T[V_3] - E_0$. Note that $\delta^0(T[V_3] - E_0) \geq |V_3| - 3$. Apply Lemma 3.4 to $T[V_3] - E_0$ with parameters $2, k-1$ corresponding to s, k , respectively. Then we obtain an ordering $\sigma_3 = (v_1, \dots, v_{|V_3|})$ and a $(\sigma_3, k-1, 2k-1)$ -good digraph $D_3 \subseteq T[V_3] - E_0$ with $|E(D_3)| \leq (k-1)|V_3| + (k-1)$. Let

$$E_3 := E(D_3), \quad W_3^- := \sigma_3(1, 2k-1) \quad \text{and} \quad W_3^+ := \sigma_3(|V_3| - 2k + 2, |V_3|).$$

This verifies (E3)₁. To verify (E3)₂, we consider a set $S \subseteq V(T)$ with $k-1$ vertices and a vertex $v \in V_3 \setminus S$. Then we consider a path P' in $D_3 - S$ with $E(P') \subseteq E(D_3)$ from v to v_i which maximizes i . If $i \geq |V_3| - 2k + 2$, then $v_i \in W_3^+$ and we are done. If $i < |V_3| - 2k + 2$, the maximality of i implies $N_{D_3}^+(v_i) \subseteq S$ by (D1) and the fact that D_3 is $(\sigma, k-1, 2k+1)$ -good. Since

$$k-1 \stackrel{(D2)}{\leq} |N_{D_3}^+(v_i)| \leq |S| = k-1,$$

we have $S = N_{D_3}^+(v_i)$. Because $v_i \in V_3$, by (5.15) there exists $s \in [3k-1]$ such that $v_i \in V^{\text{int}}(P_s)$. We let P'' be the sub-path of P_s from v_i to b_{j_s} . Since P_s is backwards-transitive, every vertex in $V(P'')$ should be in $N_T^-(v_i)$ except v_i and the second vertex, say u' , of P'' . Since $\overrightarrow{v_i u'} \in E_0$ and $E(D_3) \subseteq T[V_3] - E_0$, $u' \notin N_{D_3}^+(v_i')$. Thus

$$V(P'') \cap S \subseteq (N_T^-(v_i) \cup \{v_i, u'\}) \cap N_{D_3}^+(v_i) = \emptyset.$$

Thus P'' does not intersect with S . So $E(P') \cup E(P'')$ contains a path P in $T-S$ from v to V_1 with $E(P) \subseteq E_0 \cup E_3$. This shows (E3)₂. We can show (E3)₃ in a similar way. This proves Claim 5.6.

Claim 5.7. *There exist a set of arcs $E_4 \subseteq A(T)$ and two sets $W_4^+, W_4^- \subseteq V_4$ satisfying the following.*

- (E4)₁ $|E_4| \leq k|V_4| - k$ and $|W_4^+|, |W_4^-| \leq 2k-1$.

- (E4)₂ For a set $S \subseteq V(T)$ of size $k-1$ and a vertex $v \in V_4 \setminus S$, there exists a path P in $T-S$ from v to W_4^+ with $E(P) \subseteq E_4$.
- (E4)₃ For a set $S \subseteq V(T)$ of size $k-1$ and a vertex $v \in V_4 \setminus S$, there exists a path P in $T-S$ from W_4^+ to v with $E(P) \subseteq E_4$.

To prove Claim 5.7, we apply Lemma 3.4 to $T[V_4]$ with parameters $0, k$ corresponding to s, k , respectively. Then we obtain an ordering σ_4 and a $(\sigma_4, k, 2k-1)$ -good digraph $D_4 \subseteq T[V_4]$ with $|E(D_4)| \leq k|V_4| - k$. Let

$$E_4 := E(D_3), \quad W_4^+ := \sigma_4(|V_4| - 2k + 2, |V_4|) \text{ and } W_4^- := \sigma_4(1, 2k - 1),$$

then we have $|E_4| = |E(D_4)| \leq k|V_4| - k$, $|W_4^-| \leq 2k - 1$ and $|W_4^+| \leq 2k - 1$. Hence (E4)₁ holds. By Claim 3.1, for any $S \subseteq V(T)$ of $k-1$ vertices and $v \in V_4 \setminus S$, we can find a path P in $T[V_4] \setminus S$ from v to W_4^+ and a path P' in $T[V_4] \setminus S$ from W_4^- to v . This shows (E4)₂ and (E4)₃. This proves Claim 5.7.

We define W^+ and W^- as follows.

$$W^+ := W_2^+ \cup W_3^+ \cup W_4^+ \quad \text{and} \quad W^- := W_2^- \cup W_3^- \cup W_4^-.$$

Note that $W^+, W^- \subseteq V \setminus (A \cup B)$. Thus A in-dominates W^+ and B out-dominates W^- . Now we take E_5 as follows to make connections from W^+ to $\{a_{i_1}, \dots, a_{i_k}\}$ and from $\{b_{j_1}, \dots, b_{j_k}\}$ to W^- .

Claim 5.8. *There exists a set of arcs $E_5 \subseteq E(T)$ satisfying the following.*

- (E5)₁ $|E_5| \leq 75k^2$
- (E5)₂ For $t \in [k]$, a vertex $v \in W^+$ and a set $S \subseteq V(T) \setminus \{a_{i_t}, v\}$ of at most $k-1$ vertices, there exists a path $P(v, t)$ in $T-S$ from v to a_{i_t} such that $E(P(v, t)) \subseteq E_5$.
- (E5)₃ For $t \in [k]$, a vertex $v \in W^-$ and a set $S \subseteq V(T) \setminus \{b_{j_t}, v\}$ of at most $k-1$ vertices, there exists a path $Q(v, t)$ in $T-S$ from b_{j_t} to v such that $E(Q(v, t)) \subseteq E_5$.

By (A2) and (A3), for each $u \in W^+$ and $s \in [3k-1]$ there exists $c_{u,s} \in N_T^+(u) \cap A_s$ such that $c_{u,s} = a_s$ or $a_s \in N_T^+(c_{u,s})$. Let

$$P(u, s) := \begin{cases} (u, c_{u,s}, a_s) & \text{if } c_{u,s} \neq a_s, \\ (u, a_s) & \text{otherwise.} \end{cases}$$

Similarly, for $u \in W^-$ and $s \in [3k-1]$, there is a path $Q(u, s)$ from b_s to u with length at most 2 lying entirely in $B_s \cup \{u\}$. Let

$$E_5 := E(T[A_{\text{sink}}]) \cup E(T[B_{\text{source}}]) \cup \bigcup_{u \in W^+} \bigcup_{s=1}^{3k-1} E(P(u, s)) \cup \bigcup_{u \in W^-} \bigcup_{s=1}^{3k-1} E(Q(u, s)).$$

Then we have

$$\begin{aligned} |E_5| &\leq |E(T[A_{\text{sink}}])| + |E(T[B_{\text{source}}])| + \sum_{u \in W^+} \sum_{s=1}^{3k-1} |E(P(u, s))| + \sum_{u \in W^-} \sum_{s=1}^{3k-1} |E(Q(u, s))| \\ &\leq \binom{3k-1}{2} + \binom{3k-1}{2} + (6k-2)|W^+| + (6k-2)|W^-| \leq 75k^2. \end{aligned}$$

We get the final inequality from (E2)₁, (E3)₁ and (E4)₁. To verify (E5)₂, consider a set S of $k-1$ vertices and an index $t \in [k]$ such that $a_{i_t} \notin S$ and a vertex $v \in W^+ \setminus S$. Recall that a_{i_t} has at least k in-neighbors in A_{sink} as defined before Claim 5.1. This with the fact that A_1, \dots, A_{3k-1} are pairwise disjoint implies that there exists an index $s \in [3k-1]$ such that $a_s \in N_T^-(a_{i_t})$ and $A_s \cap S = \emptyset$. Then $P(v, s) \cup \overrightarrow{a_s a_{i_t}}$ contains a path P from v to a_{i_t} , where P does not intersect with S because P belongs to $A_s \cup \{v\} \cup \{a_{i_t}\}$. Also $E(P) \subseteq E_5$, this shows (E5)₂. We can also show (E5)₃ similarly. This proves Claim 5.8.

Now we define the desired spanning strongly k -connected digraph $D \subseteq T$. Let

$$V(D) := V(T) \quad \text{and} \quad E(D) := E_0 \cup E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5.$$

Because $\bigcup_{s=1}^k V^{\text{int}}(P_s) \subseteq V'_1 \cup V_2 \cup V_3$, we have $|E_0| \leq |V'_1| + |V_2| + |V_3| - k$. By (E1)₁, (E2)₁, (E3)₁, (E4)₁ and (E5)₁ we have

$$\begin{aligned} |E(D)| &\leq |E_0| + |E_1| + |E_2| + |E_3| + |E_4| + |E_5| \\ &\leq (|V'_1| + |V_2| + |V_3| - k) + (k|V_1| + (k-1)|V'_1| + 680k^2 \log(k+1)) + (k|V_2| - k) \\ &\quad + ((k-1)|V_3| + (k-1)) + (k|V_4| - k) + 75k^2 \\ &\leq k(|V_1| + |V'_1| + |V_2| + |V_3| + |V_4|) + |V_2| + 740k^2 \log(k+1) \\ &\stackrel{(E1)_1}{\leq} k|V| + 750k^2 \log(k+1) \end{aligned}$$

since $680k^2 \log(k+1) + 75k^2 \leq 740k^2 \log(k+1)$ for $k \geq 2$.

Now it suffices to show that D is strongly k -connected. For any set $S \subseteq V(T)$ of $k-1$ vertices and any two distinct vertices $u, v \notin S$, we claim that there is a path from u to v in $D - S$. First, since P_1, \dots, P_k are vertex-disjoint there exists $t \in [k]$ such that $V(P_t)$ does not intersect with S . We find a path P in $D - S$ from u to $u' \in W^+$ as follows:

Case 1. $u \in V_2 \cup V_4$.

There exists a path P in $D - S$ from u to $u' \in W^+$ by (E2)₂ and (E4)₂.

Case 2. $u \in V_1 \cup V'_1$.

By (E1)₂, there is a path Q in $D - S$ from u to a vertex $u_0 \in V_2$. Also (E2)₂ implies that there is a path Q' in $D - S$ from u_0 to $u' \in W^+$. Thus $E(Q) \cup E(Q')$ contains a path P in $D - S$ from u to $u' \in W^+$.

Case 3. $u \in V_3$.

By (E3)₂, there is a path R in $D - S$ from u to a vertex $u_0 \in W^+ \cup V_1$. If $u_0 \in W^+$, then let $u' = u_0$ and $P := R$. Otherwise, there is a path R' in $D - S$ from u_0 to $u' \in W^+$ by the Case 2. Thus $E(R) \cup E(R')$ contains a path P in $D - S$ from u to $u' \in W^+$.

In a similar way, we can also show that there is a path Q in $D - S$ from a vertex $v' \in W^-$ to v . By Claim 5.8, there is a path $P(u', t)$ in $D - S$ from u' to a_{it} , and a path $Q(v', t)$ in $D - S$ from b_{jt} to v' . Thus $E(P') \cup E(P(u', t)) \cup E(P_t) \cup E(Q(v', t)) \cup E(Q')$ contains a path in $D - S$ from u to v . Therefore, D is strongly k -connected.

6. CONCLUDING REMARKS

The proof of Theorem 1.3 is algorithmic. Indeed, when we use Lemma 3.4, we use Claim 3.3 to find the ordering σ and a subgraph D in polynomial-time on $n = |V(T)|$, where T is a given strongly k -connected tournament. Also when we find k -fans, we use a standard application of algorithms that find maximum-flow. We only need these tools in the proof, and the proof itself immediately gives a polynomial-time algorithm to find the desired digraph D .

In [5], Bang-Jensen, Huang and Yeo introduced an n -vertex tournament $\mathcal{T}_{n,k}$ for $n \geq k$ such that every strongly k -arc-connected spanning subgraph of $\mathcal{T}_{n,k}$ contains at least $nk + \frac{k(k-1)}{2}$ arcs. Since every strongly k -connected digraphs are also strongly k -arc-connected, this example shows that the term $750k^2 \log(k+1)$ in Theorem 1.3 is asymptotically best possible up to logarithmic terms. We finish the paper with the following natural question:

Conjecture 6.1. *For a positive integer k , there is a constant C such that every strongly k -connected n -vertex tournament T contains a strongly k -connected spanning subgraph D with at most $kn + Ck^2$ arcs.*

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